On tropical and Kapranov ranks of tropical matrices

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1 Introduction

Let us consider the tropical semiring $(\mathbf{R}, \oplus, \odot)$, where:

$$x \oplus y := min\{x, y\} \qquad x \odot y := x + y$$

For matrices with coefficients in the tropical semiring, we can define the tropical sum and the tropical product in the obvious way.
In [DSS] Develin, Santos and Sturmfels compared three possible definitions for the rank of a tropical matrix.

Definition 1 *Let $A \in M(r \times r, \mathbf{R})$. We define*

$$det(A) := \oplus_{\sigma \in S_r} a_{1,\sigma(1)} \odot .... \odot a_{r,\sigma(r)} = min_{\sigma \in S_r}\{a_{1,\sigma(1)} + ... + a_{r,\sigma(r)}\}$$

where S_r denotes the symmetric group on r elements.
Besides we say that A is singular if the minimum in $det(A)$ is attained at least twice.

Definition 2 *Let M be a matrix $m \times n$ with entries in $\mathbf{R}$. We define:*
1) (Barvinok rank) $rk_B(M) := min\{r |\ \exists A_1, ..., A_r \in M(m \times 1, \mathbf{R})$ and $B_1, ..., B_r \in M(1 \times n, \mathbf{R})$ s.t. $M = A_1 \odot B_1 \oplus ... \oplus A_r \odot B_r\}$
2) (Kapranov rank) $rk_K(M) := min\{r |\ \exists$ a tropical linear space of dimension r containing the columns of $M\}$ (see §2 for the definition of tropical linear space)
3) (Tropical rank) $rk_t(M) := max\{r |\ \exists$ a nonsingular minor $r \times r$ of $M\}$

Develin, Santos and Sturmfels proved

Theorem 3 *[DSS]. For every matrix $M \in M(m \times n, \mathbf{R})$ we have*

$$1 \leq rk_t(M) \leq rk_K(M) \leq rk_B(M) \leq min\{m, n\}$$

Besides, if either $rk_t(M)$ or $rk_K(M)$ is equal either to 1 or to 2 or to $min\{m, n\}$, then $rk_K(M) = rk_t(M)$.

Furthermore they showed that the inequalities may be strict: to show that the first inequality may be strict, they exhibit a matrix 7×7 whose tropical rank is 3 and whose Kapranov rank is 4; they wonder whether there exists a matrix 5×5 whose tropical and Kapranov ranks disagree, in particular with tropical rank 3 ad Kapranov rank strictly greater than 3 (observe that if M is a matrix 5×5 with tropical rank $1, 2, 4, 5$, then, by the theorem above and Remark 6 below, which is an obvious corollary of the theorem above, Kapranov rank is equal to the tropical rank; then the unique case for a matrix 5×5 when tropical rank and Kapranov rank may disagree is when the tropical rank is 3).
Here we prove the following theorem,

Theorem 4 *Let A be a matrix $g \times 5$ of tropical rank 3. Then also the Kapranov rank is 3.*

Finally we quote the following result by Kim and Roush:

Theorem 5 *[KR]. Kapranov rank of tropical matrices is not bounded in terms of tropical ranks, that is, given a positive integer r, it is impossible to find a bound $N(r)$ so that all matrices of tropical rank r have Kapranov rank less or equal than $N(r)$.*

2 Some recalls

Before proving our theorem, we need to recall what a linear tropical space is (we have used it in the definition of Kapranov rank) and a proposition by Develin, Santos and Sturmfels.
Let K be the algebraic closure of $\mathbf{C}(\tau)$, that is the field of Puiseux series with complex coefficients.

$$K = \{c_1\tau^{a_1} + c_2\tau^{a_2} + ...|\ c_i \in \mathbf{C},\ a_i \in \mathbf{Q},\ a_1 < a_2 < ....\ \textit{and the } a_i \textit{ have}$$
$$\textit{a common denominator}\}$$

Let us define
$$\tilde{K} = \{c_1\tau^{a_1} + c_2\tau^{a_2} + ...|\ c_i \in \mathbf{C},\ a_i \in \mathbf{R},\ a_1 < a_2 < ....\}$$
Let $ord : \tilde{K} \to \mathbf{R}$ be the map
$$(c_1\tau^{a_1} + c_2\tau^{a_2} + ...) \mapsto a_1$$
and we call ord also the obvious map $ord : \tilde{K}^d \to \mathbf{R}^d$
Let $orc : \tilde{K} \to \mathbf{R}$ be the map
$$(c_1\tau^{a_1} + c_2\tau^{a_2} + ...) \mapsto c_1$$
Let I be an ideal in $K[x_1, ..., x_d]$. Let

$$\tilde{V}(I) = \{x \in (\tilde{K} - \{0\})^d|\ f(x) = 0\ \forall f \in I\}$$

We define the tropical variety of I as $ord(\tilde{V}(I))$.
We say that a tropical variety $ord(\tilde{V}(I))$ is linear if I has a set of linear generators.

Remark 6 *Let M be a matrix $(k+1) \times n$ with $rk_t(M) = k$ with $n \geq k+1$. Then $rk_K(M) = k$.*

Proof. If $rk_K(M)$ were not k, it would be $k+1$ (because $rk_K(M) \geq rk_t(M) = k$ by the first disequality of Theorem 3), but in this case also $rk_t(M)$ would be $k+1$ by the second statement of Theorem 3. $\square$

Proposition 7 *[DSS] Let $M \in M(m \times n, \mathbf{R})$. The following are equivalent:*
1) $rk_K(M) \leq r$
2) there exists $F \in M(m \times n, \tilde{K} - \{0\})$ with $rk(F) \leq r$ and F "lift" of M, that is $ord(F) = M$.

3 Proof of the theorem

Notation 8 • *When we write a matrix, if in an entry we write $*$, this means that this entry is nonzero.*
• *U.s.r.c. means "up to swapping rows and columns".*
• *If A is a matrix, $A_{\hat{\imath},\cdot}$ is the matrix obtained from A by taking off the i-th row, $A_{(i)}$ is the i-th row and $A^{(j)}$ is the j-th column.*
• *We say that we can develop a tropical matrix A $m \times n$ by one of its rows $A_{(i)}$ with coefficients of order $l_1, ..., l_{m-1}$ if, for any lift F of $A_{\hat{\imath},\cdot}$, we can find coefficients in K of order $l_1, ..., l_{m-1}$ such that the linear combination of the rows of F with these coefficients is a lift of $A_{(i)}$.*

Remark 9 (Important remark). *Suppose we know that every matrix $(g-1) \times n$ with tropical rank s has the Kapranov rank s for any $s \leq k$. Let A be a matrix $g \times n$ with tropical rank k such that we can develop A or one of its submatrix $p \times n$ by one of its row $A_{(i)}$. Then also Kapranov rank of A is k.*
In fact there exists a lift of rank $\leq k$ of $A_{\hat{\imath},\cdot}$; this together with the linear combination of its rows given by the developing will give a lift of rank $\leq k$ of A.
We will apply this remark with $n = 5$.

Remark 10 *Let $a, b, c, d \in K$ of order 0 and $r, r' \in \mathbf{R}^+$ with $r \neq r'$. Then there exist $\lambda_1, \lambda_2 \in K$ of order greater or equal than $-m$ ($m \in \mathbf{R}^+$) such that*

$$ord(\lambda_1 a + \lambda_2 b) = r \qquad ord(\lambda_1 c + \lambda_2 d) = r'$$

iff $\begin{pmatrix} a \\ b \end{pmatrix}$ *and* $\begin{pmatrix} c \\ d \end{pmatrix}$ *are indipendent and $ord(ad - bc) \leq min\{r, r'\} + m$*

Remark 11 *Let $r \in \mathbf{N}$, $r \geq 2$. Let $X_1, ..., X_r \in K$ (and possibly $Z_1, ..., Z_s \in K$) of order respectively $x_1, ..., x_r$, $(z_1, ..., z_s)$. Let $l_1, ..., l_r, k, (m_1, ..., m_s) \in K$ such that*

$$h =: x_1 + l_1 = ... = x_r + l_r \leq k \quad (m_1 + z_1, ..., m_s + z_s > h)$$

a) Then there exist $\lambda_1, ..., \lambda_r, (\mu_1, ..., \mu_s)$ of order $l_1, ..., l_r, (m_1, ..., m_s)$ such that

$$\lambda_1 X_1 + ...\lambda_r X_r (+\mu_1 Z_1 + ... + \mu_s Z_s)$$

has order k.
b) If $k > h$, choosen $\lambda_1, ..., \lambda_{r-1}$ of order $l_1, ..., l_{r-1}$ such that $\lambda_1 X_1 + ...\lambda_{r-1} X_{r-1}$ has order h (thus if $r = 2$ for any choice of λ_1 of order l_1) (and choosen $\mu_1, ..., \mu_s$ of order $m_1, ..., m_s$), we can choose λ_r of order l_r such that

$$\lambda_1 X_1 + ...\lambda_r X_r (+\mu_1 Z_1 + ... + \mu_s Z_s)$$

has order k
c) If $k = h$, choosen in any way $\lambda_1, ..., \lambda_{r-1}$ of order $l_1, ..., l_{r-1}$, (and choosen $\mu_1, ..., \mu_s$ of order $m_1, ..., m_s$), we can choose λ_r of order l_r such that

$$\lambda_1 X_1 + ...\lambda_r X_r (+\mu_1 Z_1 + ... + \mu_s Z_s)$$

has order k.

Remark 12 *Let $Y_1, Y_2, Y_3, B_1, B_2, Z \in K$ of order respectively y_1, y_2, y_3, b, b, z. Let $t \geq z > b$. Then there exist $\lambda_1, \lambda_2 \lambda_3 \in K$ of order 0 such that*
$\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$ has order y
$\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 Z$ has order t.

Proof. Obviously if $ord(\lambda_i) = 0$ $i = 1, 2, 3$, then $\lambda_1 Y_1 + \lambda_2 Y_2 + \lambda_3 Y_3$ has order y iff $\sum_{i=1,2,3} orc(\lambda_i) orc(Y_i) \neq 0$.
Besides observe that if $ord(\lambda_i) = 0$ $i = 1, 2, 3$ and if $\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 Z$ has order t then $orc(\lambda_2) = orc(\lambda_1) \frac{orc(B_1)}{orc(B_2)}$.
So choose λ_1 and λ_3 of ord 0 such that $orc(\lambda_1) orc(Y_1) + orc(\lambda_1) \frac{orc(B_1)}{orc(B_2)} + orc(\lambda_3) orc(Y_3) \neq 0$. Then, by using the equation $ord(\lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 Z) = t$, define λ_2, for instance take $\lambda_2 = \frac{-\lambda_1 B_1 - \lambda_3 Z + \tau^t}{B_2}$ (it will have order 0).

□

Remark 13 *Let $A_1, A_2, B_2, B_3, H, Y \in K$ of order respectively a, a, b, b, h, y. Let $t, h > b$ and $z, y > a$. Then there exist $\lambda_1, \lambda_2 \lambda_3 \in K$ of order 0 such that*
$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 Y$ has order z
$\lambda_1 H + \lambda_2 B_2 + \lambda_3 B_3$ has order t.

Proof. Choose λ_1 and λ_3 such that

$$\frac{-\lambda_1 A_1 - \lambda_3 Y + \tau^z}{A_2} = \frac{-\lambda_1 H - \lambda_3 B_3 + \tau^t}{B_2}$$

Then define λ_2 as one of the two members of the above equation. □

Remark 14 *Let $A_1, A_2, B_2, B_3, B_4, H, Y, W \in K$ of order respectively a, a, b, b, b, h, y, w. Let $t, h > b$ and $z, y, w > a$. Let $c_1, c_3 c_4, c \in \mathbf{R} - \{0\}$. Then there exist $\lambda_1, \lambda_2 \lambda_3, \lambda_4 \in K$ of order 0 such that*
$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 Y + \lambda_4 W$ has order z
$\lambda_1 H + \lambda_2 B_2 + \lambda_3 B_3 + \lambda_4 B_4$ has order t
$c_1 orc(\lambda_1) + c_3 orc(\lambda_3) + c_4 orc(\lambda_4) \neq 0$
$orc(\lambda_1)/orc(\lambda_4) \neq c$.

Lemma 15 *Let A be a a matrix 5×5 of tropical rank 3. Suppose that there is a row with 5 zeroes, that every entry is greater or equal than 0, that in every column there are at least two zeroes. Then also Kapranov rank of A is 3.*

Proof. Obviously we can develop by the zero row. □

Lemma 16 *Let A be a matrix 4×5 of kind*

$$\begin{pmatrix} 0 & 0 & 0 & . & . \\ . & . & . & 0 & . \\ . & . & . & 0 & 0 \\ . & . & . & 0 & 0 \end{pmatrix}$$

with all coefficients greater or equal than 0. If the tropical rank is 3, then we can develop A by one of its row.

Proof. Observe that the tropical determinant of a matrix

$$\begin{pmatrix} 0 & 0 & 0 & . \\ . & . & . & 0 \\ . & . & . & 0 \\ . & . & . & 0 \end{pmatrix}$$

with all coefficient greater or equal than 0, is the minimum M of the sums of the couples of coefficients of the left lower submatrix 3×3 which are not in the same row and column. So, if the tropical rank is less or equal than 3, there are at least two couples of coefficients, not lined up, whose sum is M. There are two possibilities:
1) these two couples are disjoint, $a, b+h$ and $a+h, b$ let's say $h \in \mathbf{R}^{+}$, $a \leq b$.
2) these two couples have an element in common, that is there are two coefficients equal to b and one a not lined up, whose sum is M.
1) There are 3 subcases, u.s.r.c. and up to transposing,

$$1.1)\begin{pmatrix} 0 & 0 & 0 & . \\ a+h & a & . & 0 \\ . & b & . & 0 \\ b+h & . & . & 0 \end{pmatrix} \quad 1.2)\begin{pmatrix} 0 & 0 & 0 & . \\ a+h & a & . & 0 \\ . & b & . & 0 \\ . & . & b+h & 0 \end{pmatrix} \quad 1.3)\begin{pmatrix} 0 & 0 & 0 & . \\ a+h & a & . & 0 \\ b+h & b & . & 0 \\ . & . & . & 0 \end{pmatrix}$$

2) U.s.r.c. and up to transposing, we have two cases:

$$2.1)\begin{pmatrix} 0 & 0 & 0 & . \\ . & a & . & 0 \\ . & . & b & 0 \\ . & . & b & 0 \end{pmatrix} \quad 2.2)\begin{pmatrix} 0 & 0 & 0 & . \\ b & . & . & 0 \\ . & b & . & 0 \\ . & . & a & 0 \end{pmatrix}$$

with $a \leq b$
Observe that the submatrix $A_{\cdot,(1,2,3,4)}$ of A must be of one of the forms above, u.s.r.c., since its rank is less or equal than 3.
Case 1.1) Since also $A_{\cdot,(1,2,3,5)}$ has rank less or equal than 3 then, u.s.c.,

$$A = \begin{pmatrix} 0 & 0 & 0 & . & . \\ a+h & a & u & 0 & . \\ s & b & y & 0 & 0 \\ b+h & t & w & 0 & 0 \end{pmatrix} \quad or \quad A = \begin{pmatrix} 0 & 0 & 0 & . & . \\ a+h & a & u & 0 & . \\ b+h & t & w & 0 & 0 \\ s & b & y & 0 & 0 \end{pmatrix}$$

(that is $a+h$ and a must be lined up with the nonzero element among $A_{2,5}, A_{3,5}, A_{4,5}$). U.s.r., we can suppose the first.
We must have $t \geq b$, $w, s, y \geq b+h$, $u \geq a+h$. Suppose the inequalities are strict.
we can divide into many cases, we use Remark 11:

• $t < min\{w, y, u+b-a\}$
- if $y \neq u+b-a$, we can develop A by $A_{(4)}$ with coefficients of order respectively $min\{w, y, u+b-a\}, b-a, 0$
- if $y = u+b-a$:
if $w \leq y$ we can develop A by $A_{(4)}$ with coefficients of order respectively $w, b-a, 0$;
if $w > y$ we can develop $A_{(2,3,4),\cdot}$ by $A_{(3)}$ with coefficients of order respectively $b-a, 0$;

• $t \geq min\{w, y, u+b-a\}$
- if $y \neq u+b-a$, we can develop A by $A_{(4)}$ with coefficients of order respectively $min\{w, y, u+b-a\}, b-a, 0$ (use Remark 13);
- if $y = u+b-a$,
if $w \leq y$ we can develop A by $A_{(4)}$ with coefficients of order respectively $w, b-a, 0$ (if $w = y$ use Remark 12);
if $w > y$ we can develop $A_{(2,3,4),\cdot}$ by $A_{(4)}$ with coefficients of order respectively $b-a, 0$;
Analogously the other cases. □

Lemma 17 *Let A be a matrix $g \times 5$ of kind*

$$\begin{pmatrix} v' & u' & r' & 0 & 0 \\ v & u & r & 0 & 0 \\ 0 & 0 & 0 & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & . & . \end{pmatrix}$$

with all coefficients greater or equal than 0. If the tropical rank is 3, then also Kapranov rank is 3.

Proof. We can suppose that $r = min\{v, v', u, u', r', r\}$.
Choose lifts $F^{(1)}, F^{(4)}, F^{(5)}$ of $A^{(1)}, A^{(4)}, A^{(5)}$ of the kind respectively:

$$\begin{pmatrix} c\tau^{v'} \\ h\tau^{v} \\ l_1 \\ \vdots \\ l_{g-2} \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \\ . \\ \vdots \\ . \end{pmatrix} \quad \begin{pmatrix} -1+3t^{min\{v',r'\}} \\ -1+2t^{r} \\ . \\ \vdots \\ . \end{pmatrix}$$

with $c \in K$ of order 0 and $h, l_1, ..., l_{g-2} \in \mathbf{R} - \{0\}$
By Remark 10 there exist $\mu_2, \mu_3 \in K$ of order ≥ 0 such that

$$ord\left(\mu_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \mu_3 \begin{pmatrix} -1+3t^{min\{v',r'\}} \\ -1+2t^{r} \end{pmatrix}\right) = \begin{pmatrix} min\{v', u'\} \\ u \end{pmatrix}$$

Choose c such that $(F^{(1)} + F^{(4)} + F^{(5)})_1$ has *ord* equal to $A_1^{(3)}$ that is r'.
There exist $\mu_1 \in K$ of order 0 such that $(\mu_1 F^{(1)} + \mu_2 F^{(4)} + \mu_3 F^{(5)})_1$ has *ord* equal to $A_1^{(2)}$, that is u'.
Obviously we can choose $h, l_1, ..., l_{g-2}$ such that

$$ord(F^{(1)} + F^{(4)} + F^{(5)}) = A^{(3)} \quad and \quad ord(\mu_1 F^{(1)} + \mu_2 F^{(4)} + \mu_3 F^{(5)}) = A^{(2)}$$

□

Lemma 18 *Let A be a a matrix 5×5 of tropical rank 3. Suppose that there is a row with 4 zeroes, that every entry is greater or equal than 0, that in every column there are at least two zeroes and that in every row there is at least one zero. Then, if we are not in case of Lemma 17, we can develop A by the row with 4 zeroes and then also the Kapranov rank of A is 3.*

Proof. U.s.r.c. we can suppose that $A_{1,2} = ... = A_{1,5} = 0$. U.s.r, we can suppose $A_{2,1} = ... = A_{k,1} = 0$, $0 < A_{k+1,1}, ..., A_{s,1} \leq r$, $A_{s+1,1}, ..., A_{5,1} > r$ with $k \geq 3$.
We state that we can develop A by the first row with coefficients $\lambda_2, ..., \lambda_5$ of order 0, in fact:
let $(F_{i,j})_{i=2,...,4 \quad j=1,...,5}$ be a lift of $A_{[2,...,5],\cdot}$
i) to have $ord(\sum_{i=2,...,5} \lambda_i F_{i,1}) = r$, it is sufficient to choose $\lambda_2, ..., \lambda_5$ of order 0 such that

$$\sum_{i=3,...,k} orc(\lambda_i) orc(F_{i,1}) \neq 0$$

(so that $ord(\sum_{i=3,...,k} \lambda_i F_{i,1}) = 0$) and then choose

$$\lambda_2 = \frac{-\sum_{i=3,...,s} \lambda_i F_{i,1} + \tau^r}{F_{2,1}}$$

ii) to have $ord(\sum_{i=2,...,5}\lambda_i F_{i,j})=0$ for some $j\in\{2,...,5\}$, it is sufficient to choose $\lambda_2,...,\lambda_5$ of order 0 such that $\sum_{i\in\{2,...,5\}\ s.t.\ A_{i,j}=0} orc(\lambda_i)orc(F_{i,j})\neq 0$

Obviously we can easily find $\lambda_2,...,\lambda_5$ satisfying either i) and ii) unless we are in one of the following cases (u.s.r.c.)

$$A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & . \end{pmatrix}\quad A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & . & . & . \end{pmatrix}\quad A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & . & . & .\\ * & * & . & . & . \end{pmatrix}$$

In the first two cases we can conclude by Lemma 16. In the last one observe that, by assumption, we must have a zero in each of the last two rows. Observe that if

$$A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & 0 & . & .\\ * & * & * & 0 & . \end{pmatrix}$$

we would have that the tropical rank of $A_{\hat{3},\hat{5}}$ is 4, which contradicts the assumption $rk_t(A)=3$. Thus the only possibilities are (u.s.c.)

$$A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & 0 & * & *\\ * & * & 0 & * & * \end{pmatrix},\quad A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & 0 & 0 & *\\ * & * & 0 & 0 & * \end{pmatrix}\quad A=\begin{pmatrix} r & 0 & 0 & 0 & 0\\ 0 & 0 & . & . & .\\ 0 & 0 & . & . & .\\ * & * & 0 & 0 & 0\\ * & * & 0 & 0 & 0 \end{pmatrix}$$

In the last case we are in case of Lemma 17. In the first two observe that at least one of $A_{2,5}$ and $A_{3,5}$ must be zero, so we can conclude by Lemma 16. □

Proof of Theorem 4. Induction on g. The case $g=4$ follows from Remark 6. So suppose to know the statement for $g-1$. Let A be $g\times 5$. We will show that in almost all cases we can develop A or one of its submatrix $p\times 5$ by one of its rows and we will conclude by Remark 9.

U.s.r. we can suppose that the submatrix of A given by the first 4 rows, $A_{[1,..,4],\cdot}$ has tropical rank 3. Since $rk_t(A_{[1,..,4],\cdot})=3$ then also $rk_K(A_{[1,..,4],\cdot})=3$ (in fact if Kapranov rank were 4, then also the tropical rank would be 4 by Theorem 3).

So we can suppose that the columns of $A_{[1,..,4],\cdot}$ are in a tropical hyperplane. We can suppose that this hyperplane is given by $x_1\oplus...\oplus x_4$. So, up to summing some constants to the columns of A, we can suppose that in all the columns of $A_{[1,..,4],\cdot}$ we have two zero entries and all the other entries are greater or equal than 0.

If two zero entries are in the same column we say that they are twin zeroes.

For any $i=5,...,g$, up to summing a constant to all the entries of $A_{(i)}$, we can suppose also that the entries of $A_{(i)}$ are greater or equal than 0 and that the minimum is 0.

We can distinguish in five cases according to the maximum number m of the couples of twin zeroes in the same two rows of $A_{[1,2,3,4],\cdot}$.

• $m=1$.

Observe that in this case there exists a row among the first four rows of A with at least 3 zeroes (in fact if every row of $A_{[1,..,4],\cdot}$ had at most 2 zeroes, then the number of the zeroes of $A_{[1,..,4],\cdot}$ would

be less or equal than 8, but the number of the zeroes of $A_{[1,..,4],\cdot}$ is at least $2 \cdot 5 = 10$ since every columns contains at least 2 zeroes). So, u.s.r.c, we can suppose $A_{1,1} = A_{1,2} = A_{1,3} = 0$.
Since $m = 1$, A must be like this (u.s.r.c.):

$$\begin{pmatrix} 0 & 0 & 0 & a & b \\ 0 & * & * & 0 & b' \\ * & 0 & * & 0 & 0 \\ * & * & 0 & a' & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with $a, b, a', b' \neq 0$. We can develop $A_{[1,2,3,4],\cdot}$ by the first row with coefficients of order 0: if either $a' > a$ or $b' > b$, use Remark 11, otherwise use Remark 13.
• $m = 5$. It follows from Lemma 15.
• $m = 4$. It follows from Lemma 15 or Lemma 18 or Lemma 17.
• $m = 3$. U.s.r.c., we can suppose that $A_{i,j} = 0$ for $i = 1, 2$, $j = 1, 2, 3$.
By Lemmas 15, 18 and 17, we can suppose that A is like this:

$$\begin{pmatrix} 0 & 0 & 0 & v & v' \\ 0 & 0 & 0 & r & r' \\ . & . & . & . & . \\ . & . & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with $r, r', v, v' \neq 0$ and, u.s.r.c., we can suppose $r = min\{r, r', v, v'\}$.
We can distinguish in two main cases:
1) the minimum in $\{r, r', v, v'\}$ is attained at least twice.
2) the minimum in $\{r, r', v, v'\}$ is attained only once.

1) The minimum in $\{r, r', v, v'\}$ is attained at least twice.
Obviously we can suppose that $(r, r') \neq (v, v')$. If

$$A = \begin{pmatrix} 0 & 0 & 0 & v & v' \\ 0 & 0 & 0 & r & r \\ . & . & . & 0 & 0 \\ . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad or \quad A = \begin{pmatrix} 0 & 0 & 0 & v & r \\ 0 & 0 & 0 & r & r' \\ . & . & . & 0 & 0 \\ . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

we can develop $A_{[1,2,3],\cdot}$ by $A_{(2)}$ with coefficients of order 0 for the first row and r for the third row (in the second case we can suppose $v \neq r$, if not we are in the previous case).
If

$$A = \begin{pmatrix} 0 & 0 & 0 & r & v' \\ 0 & 0 & 0 & r & r' \\ . & . & . & 0 & 0 \\ . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

with $r \neq r'$, we can suppose, u.s.r.c., that $v' \geq r'$; develop $A_{[1,2,3],\cdot}$ by $A_{(2)}$ with coefficients of order 0 for the first row and r' for the third row.

2) The minimum in $\{r, r', v, v'\}$ is attained only once. Then the tropical rank of $\begin{pmatrix} 0 & v & v' \\ 0 & r & r' \\ . & 0 & 0 \end{pmatrix}$ is 3; besides observe that if we take off from A one of the first three columns, then in every of the first four rows of the so obtained matrix there are still at least two zeroes. So its rows must stay in a unique hypersurface and this hypersurface must be $x_1 \oplus x_2 \oplus x_3 \oplus x_4$. Therefore, for every $i = 5, ..., g$, in $A_{(i)}$ there are at least two zeroes in the entries $2, 3, 4, 5$, two zeroes in the entries $1, 2, 4, 5$ and two zeroes in the entries $1, 3, 4, 5$; thus, for every $i = 5, ..., g$, either $A_{(i)} = (., ., ., 0, 0)$ or in $A_{(i)}$ there at least two zeroes in the first three entries and one in the last two or $A_{(i)} = (0, 0, 0, ., .)$. If there exists $i \in \{5, ..., g\}$ such that either $A_{(i)}$ is $(., ., ., 0, 0)$ or in $A_{(i)}$ there at least two zeroes in the first three entries and one in the last two, we can conclude by Lemma 16. If, for any $i = 5, ..., g$, $A_{(i)} = (0, 0, 0, ., .)$ we can conclude by Lemma 17).

• $m = 2$.

U.s.r.c., we can suppose that $A_{i,j} = 0$ for $i, j \in \{1, 2\}$.

We can divide into many cases according how many zeroes are among the six entries in the first two lines and last three columns.

If among the six entries in the first two lines and last three columns, there were no zeroes, then A would be like this:

$$\begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ . & . & 0 & 0 & 0 \\ . & . & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and we could conclude since m should be 3.

If among the six entries in the first two lines and last three columns, there are at least three zeroes, then at least two of them must be lined up and so we can conclude by Lemma 18 and Lemma 17.

So we can suppose that among the six entries in the first two lines and last three columns, there are one or two zeroes and, in the case of two zeroes, that they are not lined up.

So we can distinguish the following cases (u.s.r.c.):

$$a)\ A = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & 0 \\ . & . & 0 & 0 & 0 \\ . & . & 0 & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad b)\ A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ . & . & 0 & . & 0 \\ . & . & . & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad c)\ A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ . & . & 0 & 0 & 0 \\ . & . & . & . & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

<u>Case a)</u> $A = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & 0 \\ . & . & 0 & 0 & 0 \\ . & . & 0 & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$

We can suppose $A = \begin{pmatrix} 0 & 0 & * & * & * \\ 0 & 0 & * & * & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$ by Lemma 18 and Lemma 16.

For any $i = 5, ..., g$, in $A_{(i)}$ there is at least a zero. If a zero is in the first four entries, then we can conclude by Lemma 16. So the unique problem is when for every $i = 5, .., g$, there is only one zero in $A_{(i)}$ and it is the last entry.

Let S_i be the submatrix $A_{[1,2,i],[3,4]}$. The tropical determinant of $A_{[124i],[2345]} = \begin{pmatrix} 0 & * & * & * \\ 0 & * & * & 0 \\ . & 0 & 0 & . \\ * & * & * & 0 \end{pmatrix}$ is the minimum in S_i. So this minumum must be attained twice.

If the minimum in S_i is attained twice in the same column, say in the first, we can develop $A_{[1,2,3,i],\cdot}$ by the first or second row with coeffcients $0, min\{A_{1,4}, A_{2,4}, A_{i,4}\}, 0$; while if the minimum s_i in S_i is attained twice but not in the same column and not in the same row, we can develop $A_{[1,2,3,i],\cdot}$ by the first or the second row with coeffcients $0, s_i, 0$

By the above remarks, we can assume that the minumum in S_i is attained twice by two entries in the same row.

Observe that if the minumum s_i in S_i is attained twice by two entries in the same row and this row is not the last one, then we can easily conclude, because, if for instance this row is $A_{(2)}$, we can develop $A_{[1,2,3,i],\cdot}$ by $A_{(2)}$ with coefficients of order $0, s_i, 0$.

So we can assume that, for every $i = 5, ..., g$, the minimum s_i in S_i is attained in the last row.

So $A = \begin{pmatrix} 0 & 0 & c & c' & y \\ 0 & 0 & d & d' & 0 \\ a & a' & 0 & 0 & 0 \\ b & b' & 0 & 0 & x \\ \cdot & \cdot & s_5 & s_5 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & s_g & s_g & 0 \end{pmatrix}$

If $c = c'$ and $d = d'$, we can conclude at once.

If $c = c'$ and $d \neq d'$, we can suppose for symmetry that $d > d'$ and we can develop $A_{\cdot,[2,3,4,5]}$ by $A^{(4)}$ with coefficients $y + d', 0, d'$.

If $d = d'$ and $c \neq c'$, we can suppose for symmetry that $c > c'$ and we can develop $A_{\cdot,[2,3,4]}$ by $A^{(3)}$ with coefficients $c', 0$ if $c' > d$, while we can develop $A_{\cdot,[2,3,4,5]}$ by $A^{(4)}$ with coefficients $c', 0, c'$ if $c' \leq d$.

So we can suppose $d \neq d'$ and $c \neq c'$

Suppose $min\{d, d', c, c'\} \in \{d, d'\}$. For symmetry we can suppose it is d. If $c = d$ or $c' = d$, we can develop $A_{\cdot,[2,3,4]}$ by $A^{(3)}$ column with coeffficients of order respectively $d, 0$. Thus we can suppose $c > d$ and $c' > d$; we can develop $A_{\cdot,[2,3,4,5]}$ by $A^{(3)}$ column with coeffficients of order respectively $min\{c, c', d + y\}, 0, d$.

Analogously if $min\{d, d', c, c'\} \in \{c, c'\}$.

Case b) We can suppose

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ . & . & 0 & * & 0 \\ . & . & * & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

if not, u.s.r.c., we are in case a). Let $R = A_{[34],[12]}$.

If all entries of R are nonzero, develop $A_{[1,2,3,4],\cdot}$ by the first row with coefficients of order $0, 0, 0$ (apply Remark 13).

If in R there are two zeroes in the same row we can conclude by Lemma 18.
If in R there are exactly two zeroes but not in the same row or column, we can suppose that $R = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$. Observe that in this case, for any $j = 1, ..., 5$, the matrix $A_{[1234],\hat{j}}$ has rank 3 and in every of its rows there are exactly two zeroes, thus for any $i = 5, ..., g$, for any $j = 1, ..., 5$, in $A_{(i),\hat{j}}$ there are at least two zeroes; thus in $A_{(i)}$ there are at least 3 zeroes; we can suppose exactly 3 by Lemma 18. So we can conclude by Lemma 16 (distinguish into the cases: at least one between $A_{i,1}$ and $A_{i,3}$ is 0, both are nonzero).
If in R there are exactly two zeroes in the same column, as above, we can suppose that for any $i = 5, ..., g$, in $A_{(i)}$ there are exactly 3 zeroes. We can conclude by Lemma 16, except if $A_{(i)} = (0, *, 0, *, 0)$ or $A_{(i)} = (0, 0, *, *, 0)$. But if, for any $i = 5, ..., g$, $A_{(i)}$ is like this, then $A^{(1)} = 0$, and then we can develp A by the first column.
Suppose in R there is only one zero; we can suppose that it is in the first column; we can observe that for any $j = 1, ..., 3$, the matrix $A_{[1234],\hat{j}}$ has rank 3 and in every of its rows there are exactly two zeroes, thus for any $i = 5, ..., g$, for any $j = 1, ..., 3$, in $A_{(i),\hat{j}}$ there are at least two zeroes. We can conclude in any case by Lemma 16, except if $A_{(i)} = (0, *, 0, 0, *)$ or $A_{(i)} = (0, 0, *, *, 0)$; in these two cases we can conclude by developping A by the first row with coefficients of order 0 by Remark 14.
<u>Case c)</u> We can suppose

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ . & . & 0 & 0 & 0 \\ . & . & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

if not, we are in case a). By Lemma 18 we can suppose also

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & 0 \\ . & . & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

If

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & 0 \\ * & * & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

we can develop $A_{[1234],\cdot}$ by the first row with coefficients $0, 0, 0$ (apply Remark 13).
So we can suppose

$$A = \begin{pmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & * & 0 & * \\ * & * & 0 & 0 & 0 \\ . & 0 & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

We can conclude by Lemma 16. $\square$

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